JOURNAL OF APPROXIMATION THEORY 59, 125–149 (1989)

Rate of Convergence of Positive Linear Operators Using an Extended Complete Tchebycheff System

GEORGE A. ANASTASSIOU

Department of Mathematical Sciences, Memphis State University, Memphis, Tennessee 38152, U.S.A.

Communicated by Oved Shisha

Received January 14, 1988

Let $[a, b] \subset \mathbb{R}$ and let $\{L_j\}_{j \in N}$ be a sequence of positive linear operators from $C^{n+1}([a, b])$ to $C([a, b])$, $n \geqslant 0$. The convergence of L_j to the unit operator I is closely related to the weak convergence of a sequence of positive finite measures μ_j to the unit measure δ_t, $t \in [a, b]$. Very general estimates with rates are given for the error $|\int_{[a,b]} f\, d\mu_j - f(t)|$, where $f \in C^{n+1}([a, b])$, in the presence of an extended complete Tchebycheff system. These lead to sharp or nearly sharp inequalities of Shisha–Mond type and are connected to the theory of best L_1 approximations by generalized polynomials.

1. INTRODUCTION

The following introductory notions come from [8], which will be of constant aid throughout this article.

Let the functions f, $u_0, u_1, ..., u_n \in C^{n+1}([a, b])$, $n \geqslant 0$, consider the Wronskians

$$W_i(x) = W[u_0(x), u_1(x), ..., u_i(x)] = \begin{vmatrix} u_0(x) & u_1(x) & \cdots & u_i(x) \\ u_0'(x) & u_1'(x) & \cdots & u_i'(x) \\ \vdots & & & \\ u_0^{(i)}(x) & u_1^{(i)} & \cdots & u_i^{(i)}(x) \end{vmatrix},$$

$$i = 0, 1, ..., n$$

and assume that all $W_i(x)$ are positive throughout $[a, b]$.

We form the functions

$$\phi_0(x) = W_0(x) = u_0(x), \qquad \phi_1(x) = \frac{W_1(x)}{(W_0(x))^2},$$

$$\phi_i(x) = \frac{W_i(x)\, W_{i-2}(x)}{(W_{i-1}(x))^2}, \qquad i = 2, 3, ..., n,$$

positive on $[a, b]$.

0021-9045/89 $3.00

Consider the linear differential operator of order $i \geqslant 1$,

$$L_i f(x) = \frac{W[u_0(x), u_1(x), ..., u_{i-1}(x), f(x)]}{W_{i-1}(x)}, \qquad i = 1, 2, ..., n+1; \quad (1)$$

also set $L_0 f(x) = f(x)$. Here $W[u_0(x), u_1(x), ..., u_{i-1}(x), f(x)]$ denotes the Wronskian of $u_0, u_1, ..., u_{i-1}, f$. Note that for $i = 1, ..., n+1$ we have

$$L_i f(x) = \phi_0(x)\,\phi_1(x) \cdots \phi_{i-1}(x) \frac{d}{dx} \frac{1}{\phi_{i-1}(x)} \frac{d}{dx} \frac{1}{\phi_{i-2}(x)} \frac{d}{dx} \cdots \frac{d}{dx} \frac{1}{\phi_1(x)} \frac{d}{dx} \frac{f(x)}{\phi_0(x)}.$$

Consider also the functions

$$g_i(x, t) = \frac{1}{W_i(x)} \cdot \begin{vmatrix} u_0(t) & u_1(t) & \cdots & u_i(t) \\ u_0'(t) & u_1'(t) & \cdots & u_i'(t) \\ \vdots & & & \\ u_0^{(i-1)}(t) & u_1^{(i-1)}(t) & \cdots & u_i^{(i-1)}(t) \\ u_0(x) & u_1(x) & \cdots & u_i(x) \end{vmatrix}, \qquad (2)$$

$$i = 1, 2, ..., n; \qquad g_0(x, t) = \frac{u_0(x)}{u_0(t)}, \qquad \text{all} \quad x, t \in [a, b].$$

Note that $g_i(x, t)$, as a function of x, is a linear combination of $u_0(x), u_1(x), ..., u_i(x)$ and furthermore

$$g_i(x, t) = \frac{\phi_0(x)}{\phi_0(t) \cdots \phi_i(t)} \int_t^x \phi_1(x_1) \int_t^{x_1} \cdots \int_t^{x_{i-2}} \phi_{i-1}(x_{i-1}) \times \int_t^{x_{i-1}} \phi_i(x_i)\, dx_i\, dx_{i-1} \cdots dx_1$$

$$= \frac{1}{\phi_0(t) \cdots \phi_i(t)} \int_t^x \phi_0(s) \cdots \phi_i(s)\, g_{i-1}(x, s)\, ds, \qquad \text{all} \quad i = 1, 2, ..., n.$$

Our work is mainly motivated by the following result (see [4, p. 376])

THEOREM. *Let* $u_0, u_1, ..., u_n \in C^n([a, b])$, $n \geqslant 0$. *Then* $\{u_i\}_{i=0}^n$ *is an extended complete Tchebysheff (E.C.T.) system on* $[a, b]$ *iff* $W_i(x)$ *are positive everywhere on* $[a, b]$, $i = 0, 1, ..., n$.

Let

$$N_n(x, t) = \int_t^x g_n(x, s)\, ds$$

and

$$E_n(x, t) = f(x) - \sum_{i=0}^{n} L_i f(t) \cdot g_i(x, t) - L_{n+1} f(t) \cdot N_n(x, t) \tag{3}$$

for all $x, t \in [a, b]$, $n \geq 0$.

Let L be a positive linear operator from $C^{n+1}([a, b])$ into $C([a, b])$, $n \geq 0$. It follows from the Riesz representation theorem that for every $t \in [a, b]$ there is a finite measure μ_t such that

$$L(f, t) = \int_{[a,b]} f(x)\, \mu_t(dx), \qquad \text{all} \quad f \in C^{n+1}([a, b]).$$

The convergence of positive linear operators to the unit operator was first studied by P. P. Korovkin in 1953 (see [5]). O. Shisha and B. Mond [7] were the first to present Korovkin's main result through an inequality giving this convergence with rates. Many others later engaged in that study (see especially [3, 6]) which also motivated our work.

Sharp general inequalities of this kind appeared for the first time in 1985 (see [1]), and the method of proof is probabilistic; there among others we find the special case of $u_i(x) = x^i$, $i = 0, 1, ..., n$. Therefore, it is still of interest to find strong upper bounds to

$$|L(f, t) - f(t)| = \left| \int_{[a,b]} f(x)\, \mu_t(dx) - f(t) \right|$$

in various important cases.

In this paper we find upper bounds to

$$\int_{[a,b]} |E_n(x, t)|\, \mu(dx); \qquad \left| \int_{[a,b]} f(x)\, \mu(dx) - f(t) \right|,$$

where μ is a positive finite measure on $[a, b]$ and t is a fixed point in $[a, b]$. These bounds lead to sharp or nearly sharp inequalities, in the natural, very general "environment" of an extended complete Tchebycheff system, for various standard cases (see Theorems 1, 2, 3). Here the convergence rates are given by the first modulus of continuity $\omega_1(L_{n+1} f, h)$, $0 < h \leq b - a$. Thus inequalities (6), (7) of Theorem 1 can be attained, i.e., they are sharp. This is seen in Theorem 1′. Furthermore, Corollaries 1 and 2 connect our results to the theory of best L_1 approximation by generalized polynomials with rates, given by strong inequalities. Equivalently, our results estimate in the very general E.C.T. setting the rate of weak convergence of a sequence of positive finite measures to the unit measure at a fixed point. At the end we give concrete examples of systems of functions $\{u_i\}_{i=0}^{n}$ satisfying the assumptions of the theorems. To the best of our knowledge this type of general theorem appears for the first time in the literature.

2. Main Results

In the following theorem we will get sharp inequalities for a particular choice of the functions u_0 and u_1.

THEOREM 1. *Let μ be a positive finite measure of mass m on $[a, b] \subset \mathbb{R}$ and t a fixed point in (a, b), such that*

$$\int_{[a,b]} |x-t|\, \mu(dx) = d > 0. \tag{4}$$

Let the functions $f(x)$, $u_0(x)$, $u_1(x)$, ..., $u_n(x)$ belong to $C^{n+1}([a, b])$, $n \geqslant 0$, and let the Wronskians $W_0(x)$, $W_1(x)$, ..., $W_n(x)$ be positive throughout $[a, b]$.

Assume that $u_0(x) = c > 0$ and $u_1(x)$ is a concave function for $x \leqslant t$ and a convex function for $x \geqslant t$. Define

$$\tilde{G}_n(x, t) = \left| \int_t^x g_n(x, s) \left\lceil \frac{|s-t|}{h} \right\rceil ds \right|, \qquad x, t \in [a, b], \tag{5}$$

where $0 < h \leqslant b-a$ is given and $\lceil \cdot \rceil$ is the ceiling of the number; $n \geqslant 0$. Assume that the first modulus of continuity $\omega_1(L_{n+1} f, h) \leqslant w$, where $w > 0$ is given.

Consider the error function

$$E_n(x, t) = f(x) - f(t) - \sum_{i=1}^{n} L_i f(t) \cdot g_i(x, t) - L_{n+1} f(t) \cdot N_n(x, t).$$

Then we have the upper bounds

$$\int_{[a,b]} |E_n(x, t)|\, \mu(dx) \leqslant w \cdot \max \left\{ \frac{\tilde{G}_n(b, t)}{b-t}, \frac{\tilde{G}_n(a, t)}{t-a} \right\} \cdot d \tag{6}$$

and

$$\begin{aligned} &\left| \int_{[a,b]} f\, d\mu - f(t) \right| \\ &\quad \leqslant |m-1|\, |f(t)| + \sum_{i=1}^{n} |L_i f(t)| \cdot \left| \int_{[a,b]} g_i(x, t)\, \mu(dx) \right| \\ &\quad\quad + |L_{n+1} f(t)| \cdot \left| \int_{[a,b]} N_n(x, t)\, \mu(dx) \right| \\ &\quad\quad + w \cdot \max \left\{ \frac{\tilde{G}_n(b, t)}{b-t}, \frac{\tilde{G}_n(a, t)}{t-a} \right\} \cdot d; \qquad n \geqslant 0. \end{aligned} \tag{7}$$

Sharpness of inequalities (6) and (7) is proved in

THEOREM 1′. *Let* $c(t)=\max(t-a, b-t)$, *where* $t\in(a,b)$ *is fixed and let* $0<h\leqslant b-a$. *For*

$$k=0, 1, \ldots, \left\lceil \frac{c(t)}{h} \right\rceil - 1$$

and $N\geqslant 1$ *define the continuous function* f_N *as follows*:

$$f_N(y)=\begin{cases} \dfrac{Nwy}{2h}+kw\left(1-\dfrac{N}{2}\right), & \text{if} \quad kh\leqslant y\leqslant\left(k+\dfrac{2}{N}\right)h; \\ (k+1)w, & \text{if} \quad \left(k+\dfrac{2}{N}\right)h<y\leqslant(k+1)h; \\ \left\lceil \dfrac{c(t)}{h}\right\rceil w, & \text{if} \quad \left(\left\lceil\dfrac{c(t)}{h}\right\rceil-1+\dfrac{2}{N}\right)h<y\leqslant c(t). \end{cases} \tag{8}$$

Observe that

$$\lim_{N\to+\infty} f_N(y)=\left\lceil \frac{y}{h}\right\rceil w, \qquad 0\leqslant y\leqslant c(t).$$

Define

$$\tilde{G}_{nN}(x,t)=\left|\int_t^x g_n(x,s)\, f_N(|s-t|)\, ds\right|, \quad \text{all} \quad x, t\in[a,b], \quad n\geqslant 0, \quad N\geqslant 1. \tag{9}$$

Then (*as* $N\to+\infty$) *inequalities* (6) *and* (7) *of Theorem* 1 *are attained, i.e., they are sharp.*

Namely:

(i) *Assume that*

$$\frac{\tilde{G}_n(b,t)}{b-t}\geqslant\frac{\tilde{G}_n(a,t)}{t-a} \qquad \text{and} \qquad d\leqslant m(b-t).$$

The optimal elements are the function

$$f(x)=\begin{cases} \tilde{G}_{nN}(x,t), & t\leqslant x\leqslant b, \\ 0, & a\leqslant x\leqslant t, \end{cases}$$

with $\omega_1(L_{n+1}f, h)\leqslant w$, *and* μ *which is the positive measure of mass* m *with masses* $[m-(d/b-t)]$ *and* $(d/b-t)$ *at* t *and* b, *respectively.*

(ii) *Assume that*

$$\frac{\tilde{G}_n(b, t)}{b-t} \leqslant \frac{\tilde{G}_n(a, t)}{t-a} \qquad \text{and} \qquad d \leqslant m(t-a).$$

The optimal elements are the function

$$f(x) = \begin{cases} 0, & t \leqslant x \leqslant b, \\ \tilde{G}_{nN}(x, t), & a \leqslant x \leqslant t, \end{cases}$$

with $\omega_1(L_{n+1} f, h) \leqslant w$, and μ which is the positive measure of mass m with masses $[m-(d/t-a)]$ and $(d/t-a)$ at t and a, respectively.

The next result relates to best L_1-approximation by generalized polynomials.

COROLLARY 1. *Inequality* (6) *of Theorem* 1 *implies*

$$\min_{\substack{(c_0, c_1, \ldots, c_n, c_{n+1}) \\ c_i \in \mathbb{R}, i=0,1,\ldots,n+1}} \int_{[a,b]} \left| f(x) - \sum_{i=0}^{n} c_i g_i(x, t) - c_{n+1} N_n(x, t) \right| \cdot \mu(dx)$$

$$\leqslant w \cdot \max \left\{ \frac{\tilde{G}_n(b, t)}{b-t}, \frac{\tilde{G}_n(a, t)}{t-a} \right\} \cdot d, \qquad n \geqslant 0. \tag{10}$$

Remark 1. Given that $d = \int_{[a,b]} |x-t| \, \mu(dx) < \infty$, where μ is a positive nonfinite measure on $[a, b]$, inequality (6) of Theorem 1 and inequality (10) are still valid.

In general we get

THEOREM 2. *Let μ be a positive finite measure of mass m on $[a, b] \subset \mathbb{R}$ and t a fixed point in $[a, b]$, such that*

$$\left(\int_{[a,b]} |x-t|^{n+2} \, \mu(dx) \right)^{1/(n+2)} = h, \tag{11}$$

where $0 < h \leqslant b-a$ is given, $n \geqslant 0$.

Let the functions $f(x)$, $u_0(x)$, $u_1(x)$, ..., $u_n(x)$ belong to $C^{n+1}([a, b])$ and let the Wronskians $W_0(x)$, $W_1(x)$, ..., $W_n(x)$ be positive throughout $[a, b]$. Assume that the first modulus of continuity $\omega_1(L_{n+1} f, h) \leqslant w$, where $w > 0$ is given.

Consider the error function

$$E_n(x, t) = f(x) - \sum_{i=0}^{n} L_i f(t) \cdot g_i(x, t) - L_{n+1} f(t) \cdot N_n(x, t).$$

Then we have the upper bounds

$$\int_{[a,b]} |E_n(x,t)|\,\mu(dx)$$
$$\leqslant w\cdot(m^{1/(n+2)}+1)\cdot\left(\int_{[a,b]} |N_n(x,t)|^{(n+2/n+1)}\,\mu(dx)\right)^{(n+1/n+2)} \tag{12}$$

and

$$\left|\int_{[a,b]} f\,d\mu - f(t)\right|$$
$$\leqslant |f(t)|\cdot\left|\int_{[a,b]} g_0(x,t)\,\mu(dx)-1\right|$$
$$+\sum_{i=1}^{n} |L_i f(t)|\cdot\left|\int_{[a,b]} g_i(x,t)\,\mu(dx)\right|$$
$$+|L_{n+1}f(t)|\cdot\left|\int_{[a,b]} N_n(x,t)\,\mu(dx)\right|$$
$$+w\cdot(m^{1/(n+2)}+1)$$
$$\cdot\left(\int_{[a,b]} |N_n(x,t)|^{(n+2/n+1)}\cdot\mu(dx)\right)^{(n+1/n+2)},\qquad n\geqslant 0. \tag{13}$$

A more general connection to best L_1 approximation by generalized polynomials is as follows:

COROLLARY 2. *Inequality* (12) *of Theorem* 2 *implies*

$$\min_{\substack{(c_0,c_1,\ldots,c_n,c_{n+1})\\ c_i\in\mathbb{R},\, i=0,1,\ldots,n+1}} \int_{[a,b]} \left|f(x)-\sum_{i=0}^{n} c_i g_i(x,t) - c_{n+1}N_n(x,t)\right|\cdot\mu(dx)$$
$$\leqslant w\cdot(m^{1/(n+2)}+1)\cdot\left(\int_{[a,b]} |N_n(x,t)|^{(n+2/n+1)}\cdot\mu(dx)\right)^{(n+1/n+2)},\ n\geqslant 0. \tag{14}$$

The next theorem improves Theorem 2 under a Lipschitz condition.

THEOREM 3. *Let μ be a positive finite measure of mass m on $[a,b]\subset\mathbb{R}$ and t a fixed point in $[a,b]$, such that*

$$\left(\int_{[a,b]} |x-t|^{n+2}\cdot\mu(dx)\right)^{1/(n+2)} = h, \tag{15}$$

where $0<h\leqslant b-a$ is given, $n\geqslant 0$.

Let the functions $f(x)$, $u_0(x), u_1(x), \ldots, u_n(x)$ *belong to* $C^{n+1}([a,b])$ *and let the Wronskians* $W_0(x), W_1(x), \ldots, W_n(x)$ *be positive throughout* $[a,b]$. *Assume that the first modulus of continuity* $\omega_1(L_{n+1}f, \delta) \leqslant A\delta^\alpha$, *all* $0 < \delta \leqslant b-a$, $A > 0$, $0 < \alpha \leqslant 1$.

Consider the error function

$$E_n(x,t) = f(x) - \sum_{i=0}^{n} L_i f(t) \cdot g_i(x,t) - L_{n+1} f(t) \cdot N_n(x,t).$$

Then we find the upper bounds $(n \geqslant 0)$

$$\int_{[a,b]} |E_n(x,t)| \, \mu(dx)$$

$$\leqslant \begin{cases} A \cdot h^\alpha \cdot \left(\int_{[a,b]} |N_n(x,t)|^{(n+2/n+1)} \cdot \mu(dx) \right)^{(n+1/n+2)}, \\ \qquad m \leqslant 1; \\ A \cdot h^\alpha \cdot m^{(1-\alpha/n+2)} \cdot \left(\int_{[a,b]} |N_n(x,t)|^{(n+2/n+1)} \cdot \mu(dx) \right)^{(n+1/n+2)}, \\ \qquad m \geqslant 1. \end{cases} \tag{16}$$

Remark 2. We see that when $\omega_1(L_{n+1}f, \delta) \leqslant A\delta^\alpha$, inequality (16) improves the corresponding results from inequalities (12) and (13) of Theorem 2.

3. Examples

(1) The system of functions $u_i(x) = x^i$, $i = 0, 1, \ldots, n$, defined on $[a,b]$, satisfies the assumptions of Theorems 1, 2.

In particular $L_i f(t) = f^{(i)}(t)$, $g_i(x,t) = (x-t)^i/i!$, $t \in [a,b]$ (see [8, p. 133]).

(2) According to [8, p. 135] consider $\phi_0(x) = 1$, $\phi_i(x) = \cosh ix$, $i = 1, \ldots, n$ defined on $[a,b]$, $t = 0 \in (a,b)$.

Note that $\phi_i(0) = 1$, $i = 0, 1, \ldots, n$, $g_0(x,s) = 1$ and

$$g_i(x,0) = \int_0^x \phi_1(s) \cdots \phi_i(s) \, g_{i-1}(x,s) \, ds, \qquad i = 1, \ldots, n.$$

In particular $g_1(x,0) = \sinh x$. Thus the system of functions $u_i(x) = g_i(x,0)$, $i = 0, 1, \ldots, n$ satisfies the assumptions of Theorems 1, 2.

Indeed, $u_0(x)=1$, $u_1(x)=\sinh x$, and clearly $u_1(x)$ is a concave function for $x\leqslant 0$, and a convex function for $x\geqslant 0$.

(3) The system of functions

$$\{u_i(x)\}_{i=0}^{n}=\{1, (-1)^{i-1}\sin ix, (-1)^i \cos ix\}_{i=1}^{(n/2)}$$

defined on $[a, b]$, $t=0\in(a, b)$, n even, satisfies the assumptions of Theorem 2.

In particular (see [8, p. 151(11)])

$$g_{2i}(x, 0)=\frac{2^i}{(2i)!}[1-\cos x]^i,$$

$$g_{2i+1}(x, 0)=\frac{2^i}{(2i+1)!}[1-\cos x]^i \sin x$$

and

$$L_{2i+1}=D(D^2+1^2)(D^2+2^2)\cdots(D^2+i^2),$$

$$L_{2i+2}f(0)=D^2(D^2+1^2)(D^2+2^2)\cdots(D^2+i^2)f(0),$$

where D indicates the operation of differentiation.

(4) Let $\phi_0(x)=1$, $\phi_i(x)=e^{\varphi(i)\cdot x}$, $i=1, ..., n$ be defined on $[a, b]$, with $\varphi(i)\neq 0$, e.g., $\varphi(i)=i$, $\varphi(i)=-i^{-1}$.

Then (see [8, p. 135]) we have

$$g_i(x, t)=\frac{1}{\phi_0(t)\cdots\phi_i(t)}\int_t^x \phi_0(s)\cdots\phi_i(s)\, g_{i-1}(x, s)\, ds, \qquad i=1, ..., n,$$

$$g_0(x, t)=1, \qquad t\in[a, b].$$

From the same reference we get that the system of functions $u_i(x)=g_i(x, t)$, $i=0, 1, ..., n$ satisfies the assumption of Theorem 2.

4. Auxiliary Results

The next results are of independent interest.

Lemma 1. *Let g be a differentiable real-valued function on $[a, b]^2\subset\mathbb{R}^2$ with $g(x, x)=0$ for all $x\in[a, b]$, and let φ be a bounded measurable real-valued function on $[a, b]$.*

Define

$$G(x, t)=\int_t^x g(x, s)\,\varphi(s)\, ds, \qquad all \quad x, t\in[a, b].$$

Then

$$\frac{\partial G(x,t)}{\partial x}=\int_t^x \frac{\partial g(x,s)}{\partial x}\,\varphi(s)\,ds.$$

Proof. Easy. ∎

As a consequence we get

LEMMA 2. *Let*

$$G_n(x,t)=\int_t^x g_n(x,s)\left\lceil\frac{|s-t|}{h}\right\rceil ds, \qquad \textit{all} \quad x,t\in[a,b],$$

where $0<h\leqslant b-a$ *and* $\lceil\cdot\rceil$ *is the ceiling of the number.*
Then

$$\frac{\partial G_n(x,t)}{\partial x}=\int_t^x \frac{\partial g_n(x,s)}{\partial x}\left\lceil\frac{|s-t|}{h}\right\rceil ds, \qquad n\geqslant 1, \tag{17}$$

and

$$\frac{\partial^2 G_n(x,t)}{\partial x^2}=\int_t^x \frac{\partial^2 g_n(x,s)}{\partial x^2}\left\lceil\frac{|s-t|}{h}\right\rceil ds, \qquad n\geqslant 2. \tag{18}$$

Proof. See [8, p. 132(6)] and apply Lemma 1 once/twice. ∎

The last result is used in

LEMMA 3. *Assume that* $u_0(x)=c>0$ *and* $u_1(x)$ *is a convex function for* $x\geqslant t$. *Let*

$$G_n(x,t)=\int_t^x g_n(x,s)\left\lceil\frac{|s-t|}{h}\right\rceil ds, \textit{ all } x,t\in[a,b], \textit{ where } 0<h\leqslant b-a,\ n\geqslant 0.$$

Then $G_n(x,t)>0$ *for* $x>t$, , $G_n(t,t)=0$, *and, as a function of* x, $G_n(x,t)$ *is strictly increasing in* $x\geqslant t$ *and continuous in* $[a,b]$.

Moreover, $G_n(x,t)$ *is a strictly convex function in* $x\geqslant t$, $n\geqslant 1$, *and* $G_0(x,t)$ *is a convex function in* $x\geqslant t$.

Proof. From $W_0(x)=\phi_0(x)=u_0(x)=c>0$ and $W_1(x)=W[u_0(x),u_1(x)]=cu_1'(x)>0$, $u_1(x)$ is a strictly increasing function everywhere on $[a,b]$. Hence $\phi_1(x)=W_1(x)/(W_0(x))^2=u_1'(x)/c>0$.

By assumption $u_1(x)$ is a convex function in $x\geqslant t$ implying that $u_1'(x)$ is an increasing function there; that is, $\phi_1(x)$ is increasing in $x\geqslant t$.

Recall that

$$g_n(x, t) = \frac{1}{\phi_1(t) \cdots \phi_n(t)} \int_t^x \phi_1(x_1) \int_t^{x_1} \cdots \int_t^{x_{n-2}} \phi_{n-1}(x_{n-1})$$
$$\times \int_t^{x_{n-1}} \phi_n(x_n)\, dx_n\, dx_{n-1} \cdots dx_1$$

and $g_n(x, t) > 0$, $(x > t)$, $g_n(t, t) = 0$; $n \geqslant 1$, with $g_0(x, t) = 1$.

Consequently

$$\frac{\partial g_n(x, t)}{\partial x} = \frac{\phi_1(x)}{\phi_1(t) \cdots \phi_n(t)} \int_t^x \phi_2(x_1) \cdots \int_t^{x_{n-2}} \phi_n(x_{n-1})\, dx_{n-1} \cdots dx_1.$$

From $\phi_i(x) > 0$, $i = 1, ..., n$, $n \geqslant 2$, and $\phi_1(x)$ being an increasing function we have that $\partial g_n(x, t)/\partial x$ is a strictly increasing function in $x \geqslant t$; note that

$$\frac{\partial g_n(x, t)}{\partial x} > 0 \ (x > t), \qquad \frac{\partial g_n(t, t)}{\partial x} = 0.$$

Thus $g_n(x, t)$ is a strictly convex function in $x \geqslant t$, $n \geqslant 2$ and clearly $g_1(x, t)$ is convex in $x \geqslant t$. One can easily prove that $G_n(x, t)$ is a continuous function in $x \in [a, b]$, $n \geqslant 0$.

From Lemma 2

$$\frac{\partial^i G_n(x, t)}{\partial x^i} = \int_t^x \frac{\partial^i g_n(x, s)}{\partial x^i} \left\lceil \frac{s - t}{h} \right\rceil ds \qquad (x \geqslant t, n \geqslant 2), \quad i = 1, 2.$$

It is clear that $G_n(x, t)$ is a strictly increasing function in $x \geqslant t$, $n \geqslant 2$.

By strict convexity of $g_n(x, s)$ in $x \geqslant s$ we get $\partial^2 g_n(x, s)/\partial x^2 > 0$ $(x > s)$, which leads to

$$\frac{\partial^2 G_n(x, t)}{\partial x^2} > 0 \ (x > t), \qquad \frac{\partial^2 G_n(t, t)}{\partial x^2} = 0.$$

Hence $G_n(x, t)$ is a strictly convex function in $x \geqslant t$, $n \geqslant 2$.

Since $g_0(x, t) = 1$, all $x, t \in [a, b]$, one has

$$G_0(x, t) = \int_t^x \left\lceil \frac{s - t}{h} \right\rceil ds \qquad (x \geqslant t).$$

Since $G_0(x, t)$ is the integral of an increasing function, it is a convex function in $x \geqslant t$; it is also strictly increasing in $x \geqslant t$. Note that

$$g_1(x, t) = \phi_1^{-1}(t) \int_t^x \phi_1(s)\, ds.$$

From $\partial g_1(x, t)/\partial x = \phi_1(x)/\phi_1(t)$ and since ϕ_1 is an increasing function, we have that $\partial g_1(x, t)/\partial x$ is increasing in $x \geqslant t$. Obviously, $\partial g_1(x, t)/\partial x > 0$ for all $x \in [a, b]$.

Let s be such that $t \leqslant s \leqslant x_1 < x_2$. Then

$$\frac{\partial g_1(x_2, s)}{\partial x}\left\lceil\frac{s-t}{h}\right\rceil \geqslant \frac{\partial g_1(x_1, s)}{\partial x}\left\lceil\frac{s-t}{h}\right\rceil.$$

Adding

$$\int_t^{x_1} \frac{\partial g_1(x_2, s)}{\partial x}\left\lceil\frac{s-t}{h}\right\rceil ds \geqslant \int_t^{x_1} \frac{\partial g_1(x_1, s)}{\partial x}\left\lceil\frac{s-t}{h}\right\rceil ds$$

and

$$\int_{x_1}^{x_2} \frac{\partial g_1(x_2, s)}{\partial x}\left\lceil\frac{s-t}{h}\right\rceil ds > 0,$$

one has

$$\int_t^{x_2} \frac{\partial g_1(x_2, s)}{\partial x}\left\lceil\frac{s-t}{h}\right\rceil ds > \int_t^{x_1} \frac{\partial g_1(x_1, s)}{\partial x}\left\lceil\frac{s-t}{h}\right\rceil ds.$$

The last inequality and Lemma 2(17) imply that $\partial G_1(x, t)/\partial x$ is strictly increasing in $x \geqslant t$, which in turn implies that $G_1(x, t)$ is a strictly convex function in $x \geqslant t$.

Since

$$\frac{\partial G_1(x, t)}{\partial x} > 0 \quad (x > t), \qquad \frac{\partial G_1(t, t)}{\partial x} = 0$$

we conclude that $G_1(x, t)$ is a strictly increasing function in $x \geqslant t$. ∎

The counterpart of Lemma 3 is as follows:

LEMMA 4. *Assume that* $u_0(x) = c > 0$ *and* $u_1(x)$ *is a concave function for* $x \leqslant t$. *When* $x \leqslant t$, $x, t \in [a, b]$, *and we have*

$$G_n(x, t) = \int_t^x g_n(x, s)\left\lceil\frac{t-s}{h}\right\rceil ds, \qquad \text{where} \quad 0 < h \leqslant b-a, \quad n \geqslant 0.$$

If n *is odd, then, as a function of* x, $G_n(x, t)$ *is a strictly decreasing and a strictly convex function in* $x \leqslant t$; *moreover,* $G_n(x, t) > 0$ *for* $x < t$. *If* n *is even, then* $G_n(x, t)$ *is a strictly increasing and a strictly concave function in* $x \leqslant t$. *Furthermore,* $G_0(x, t)$ *is a strictly increasing and a concave function in* $x \leqslant t$. *Also* $G_n(x, t) < 0$ $(x < t)$ *for* n *zero or even, and* $G_n(t, t) = 0$ *for all* $n \geqslant 0$.

Proof. By assumption $u_1(x)$ is a concave function in $x \leqslant t$ implying that $u_1'(x)$ is a decreasing function there; $\phi_1(x)$ is decreasing in $x \leqslant t$, We see that for $n \geqslant 1$

$$\frac{\partial g_n(x, t)}{\partial x} = \frac{\phi_1(x)}{\phi_1(t) \cdots \phi_n(t)} \int_t^x \phi_2(x_1) \int_t^{x_1} \cdots \int_t^{x_{n-2}} \phi_n(x_{n-1})\, dx_{n-1} \cdots dx_1$$

$$= (-1)^{n-1} \frac{\phi_1(x)\, B(x, t)}{\phi_1(t) \cdots \phi_n(t)},$$

where

$$B(x, t) = \int_x^t \phi_2(x_1) \int_{x_1}^t \cdots \int_{x_{n-2}}^t \phi_n(x_{n-1})\, dx_{n-1} \cdots dx_1 > 0 \qquad (x < t),$$

$$B(t, t) = 0.$$

Since $B(x, t)$ is a strictly decreasing function in $x \leqslant t$, we get that also $\phi_1(x) \cdot B(x, t)$ is strictly decreasing in $x \leqslant t$.

When $n > 1$ is odd

$$\frac{\partial g_n(x, t)}{\partial x} > 0 \ (x < t), \qquad \frac{\partial g_n(t, t)}{\partial x} = 0$$

and it is a strictly decreasing function in $x \leqslant t$. When n is even

$$\frac{\partial g_n(x, t)}{\partial x} < 0 \ (x < t), \qquad \frac{\partial g_n(t, t)}{\partial x} = 0$$

and it is a strictly increasing function in $x \leqslant t$.

We have proved that for n odd, $g_n(x, t) < 0$ $(x < t)$, $g_n(t, t) = 0$, and $g_n(x, t)$ is strictly concave in $x \leqslant t$ for $n > 1$; clearly $g_1(x, t)$ is concave in $x \leqslant t$. Also for n even $g_n(x, t) > 0$ $(x < t)$, $g_n(t, t) = 0$, and $g_n(x, t)$ is strictly convex in $x \leqslant t$.

From Lemma 2,

$$\frac{\partial^i G_n(x, t)}{\partial x^i} = \int_t^x \frac{\partial^i g_n(x, s)}{\partial x^i} \left\lceil \frac{t-s}{h} \right\rceil ds \qquad (x \leqslant t, n \geqslant 2), \quad i = 1, 2.$$

It is clear that if $n > 2$ is odd, then $G_n(x, t)$ is strictly decreasing and strictly convex in $x \leqslant t$, and if n is even, then $G_n(x, t)$ is strictly increasing and strictly concave in $x \leqslant t$. Note that for $n \geqslant 1$ odd, $G_n(x, t) > 0$ and for n zero or even, $G_n(x, t) < 0$, where $x < t$; moreover, $G_n(t, t) = 0$ all $n \geqslant 0$.

One can easily see that, as a function of x,

$$G_0(x, t) = \int_t^x \left\lceil \frac{t-s}{h} \right\rceil ds$$

is a concave and a strictly increasing function in $x \leqslant t$.

From $\partial g_1(x,t)/\partial x = \phi_1(x)/\phi_1(t)$ and ϕ_1 a decreasing function, we have that $\partial g_1(x,t)/\partial x$ is decreasing in $x \leqslant t$. Obviously $\partial g_1(x,t)/\partial x > 0$ for all $x \in [a, b]$.

Let s be such that $x_1 < x_2 \leqslant s \leqslant t$. Then

$$\frac{\partial g_1(x_1, s)}{\partial x} \left\lceil \frac{t-s}{h} \right\rceil \geqslant \frac{\partial g_1(x_2, s)}{\partial x} \left\lceil \frac{t-s}{h} \right\rceil.$$

Adding

$$\int_{x_2}^{t} \frac{\partial g_1(x_1, s)}{\partial x} \left\lceil \frac{t-s}{h} \right\rceil ds \geqslant \int_{x_2}^{t} \frac{\partial g_1(x_2, s)}{\partial x} \left\lceil \frac{t-s}{h} \right\rceil ds$$

and

$$\int_{x_1}^{x_2} \frac{\partial g_1(x_1, s)}{\partial x} \left\lceil \frac{t-s}{h} \right\rceil ds > 0$$

one has

$$\int_{x_1}^{t} \frac{\partial g_1(x_1, s)}{\partial x} \left\lceil \frac{t-s}{h} \right\rceil ds > \int_{x_2}^{t} \frac{\partial g_1(x_2, s)}{\partial x} \left\lceil \frac{t-s}{h} \right\rceil ds$$

or

$$\int_{t}^{x_1} \frac{\partial g_1(x_1, s)}{\partial x} \left\lceil \frac{t-s}{h} \right\rceil ds < \int_{t}^{x_2} \frac{\partial g_1(x_2, s)}{\partial x} \left\lceil \frac{t-s}{h} \right\rceil ds.$$

The last inequality and Lemma 2(17) imply that $\partial G_1(x,t)/\partial x$ is strictly increasing in $x \leqslant t$, which means that $G_1(x,t)$ is a strictly convex function in $x \leqslant t$. Since

$$\frac{\partial G_1(x,t)}{\partial x} < 0 \ (x < t), \qquad \frac{\partial G_1(t,t)}{\partial x} = 0$$

we conclude that $G_1(x,t)$ is a strictly decreasing function in $x \leqslant t$. ∎

From Lemmas 3 and 4 we obtain

LEMMA 5. *Assume that $u_0(x) = c > 0$ and $u_1(x)$ is a concave function for $x \leqslant t$ and a convex function for $x \geqslant t$.*

Let $\tilde{G}_n(x,t) = |G_n(x,t)|$, where

$$G_n(x,t) = \int_t^x g_n(x,s) \left\lceil \frac{|s-t|}{h} \right\rceil ds, \quad \text{all } x, t \in [a,b], 0 < h \leqslant b-a, n \geqslant 0.$$

Then for $n \geqslant 1$, *and as a function of* x, $\tilde{G}_n(x, t)$ *is strictly decreasing in* $x \leqslant t$ *and strictly increasing in* $x \geqslant t$; *moreover, it is continuous and strictly convex function in* $x \in [a, b]$.

$\tilde{G}_0(x, t)$ *possesses all the above properties, with the exception that it is merely a convex function in* $x \in [a, b]$. *In particular,* $\tilde{G}_n(x, t) > 0$ *for* $x \neq t$, *with* $\tilde{G}_n(t, t) = 0$, *all* $n \geqslant 0$.

Lemma 5 implies the next lemma, which is used in the proof of Theorem 1.

LEMMA 6. *Under the assumptions of Lemma* 5, *for fixed* $t \in (a, b)$, *we have that*

$$\tilde{G}_n(x, t) \leqslant \max\left\{\frac{\tilde{G}_n(b, t)}{b-t}, \frac{\tilde{G}_n(a, t)}{t-a}\right\} \cdot |x-t|, \tag{19}$$

all $x \in [a, b]$, *for all* $n \geqslant 1$.

Equality can be true only at $x = t$ *and at* $x = a$ *or* b.

The above inequality is also true for $n = 0$, *but equality can hold elsewhere, not only at the points* t, a, *or* b.

Proof. When $t < x < b$ by strict convexity of $\tilde{G}_n(x, t)$, $n \geqslant 1$, we get

$$\frac{\tilde{G}_n(x, t)}{x-t} < \frac{\tilde{G}_n(b, t)}{b-t} \qquad (\tilde{G}_n(t, t) = 0).$$

Thus

$$\tilde{G}_n(x, t) < \left(\frac{\tilde{G}_n(b, t)}{b-t}\right) \cdot (x-t) \leqslant \max\left\{\frac{\tilde{G}_n(b, t)}{b-t}, \frac{\tilde{G}_n(a, t)}{t-a}\right\} \cdot (x-t).$$

An when $a < x < t$, again by strict convexity of $\tilde{G}_n(x, t)$ we get

$$\frac{\tilde{G}_n(a, t)}{a-t} < \frac{\tilde{G}_n(x, t)}{x-t}.$$

Thus

$$\tilde{G}_n(x, t) < \left(\frac{\tilde{G}_n(a, t)}{t-a}\right) \cdot (t-x) \leqslant \max\left\{\frac{\tilde{G}_n(b, t)}{b-t}, \frac{\tilde{G}_n(a, t)}{t-a}\right\} \cdot (t-x). \quad \blacksquare$$

The next result is used in the proof of Theorem 3.

LEMMA 7. *Let μ be a positive finite measure of mass $m \leqslant 1$ on $[a, b] \subset \mathbb{R}$ and t a fixed point in $[a, b]$. Then*

$$\left(\int_{[a,b]} |x-t|^r \, \mu(dx)\right)^{1/r}$$

is an increasing function in $r > 0$.

Proof. Similar to the proof of the related result in [2, p. 155(c)] ∎

5. PROOFS OF MAIN RESULTS

Proof of Theorem 1. From [8, p. 138, Theorem II] we have

$$f(x) = f(t) + \sum_{i=1}^{n} L_i f(t) \cdot g_i(x, t) + \int_t^x g_n(x, s) \cdot L_{n+1} f(s) \, ds,$$

all $x \in [a, b]$, fixed $t \in (a, b)$, $n \geqslant 0$. And from (3) we see that

$$f(x) = f(t) + \sum_{i=1}^{n} L_i f(t) \cdot g_i(x, t) + L_{n+1} f(t) \cdot N_n(x, t)$$

$$+ \int_t^x g_n(x, s) \cdot (L_{n+1} f(s) - L_{n+1} f(t)) \, ds.$$

Thus

$$E_n(x, t) = \int_t^x g_n(x, s) \cdot (L_{n+1} f(s) - L_{n+1} f(t)) \, ds, \qquad n \geqslant 0.$$

Since $\omega_1(L_{n+1} f, h) \leqslant w$ (from [1, p. 251]), Corollary 2.2 we have

$$|L_{n+1} f(s) - L_{n+1} f(t)| \leqslant w \cdot \left\lceil \frac{|s-t|}{h} \right\rceil.$$

In the proofs of Lemmas 3, 4 we find

$$\begin{array}{lll} g_n(x, t) > 0, & x > t; & n \geqslant 1, \\ g_n(x, t) < 0, & x < t; & n \text{ odd}, \\ g_n(x, t) > 0, & x < t; & n \text{ even}, \\ g_n(t, t) = 0, & n \geqslant 1 & \text{and } g_0(x, t) = 1. \end{array}$$

Let $x \leqslant t$ and n even. Then

$$|E_n(x, t)| = \left| \int_x^t g_n(x, s) \cdot (L_{n+1} f(s) - L_{n+1} f(t)) \cdot ds \right|$$
$$\leqslant \int_x^t g_n(x, s) \cdot |L_{n+1} f(s) - L_{n+1} f(t)| \cdot ds$$
$$\leqslant w \cdot \int_x^t g_n(x, s) \cdot \left\lceil \frac{|s-t|}{h} \right\rceil \cdot ds$$
$$= w \cdot \left| \int_t^x g_n(x, s) \cdot \left\lceil \frac{|s-t|}{h} \right\rceil \cdot ds \right|.$$

That is,

$$|E_n(x, t)| \leqslant w \cdot \tilde{G}_n(x, t)$$

for $x \leqslant t$ and n even.

Let $x \leqslant t$ and n odd. Then

$$|E_n(x, t)| = \left| \int_x^t (-g_n(x, s)) \cdot (L_{n+1} f(s) - L_{n+1} f(t)) \cdot ds \right|$$
$$\leqslant \int_x^t (-g_n(x, s)) \cdot |L_{n+1} f(s) - L_{n+1} f(t)| \cdot ds$$
$$\leqslant w \cdot \int_x^t (-g_n(x, s)) \cdot \left\lceil \frac{|s-t|}{h} \right\rceil \cdot ds$$
$$= w \cdot \left| \int_t^x g_n(x, s) \cdot \left\lceil \frac{|s-t|}{h} \right\rceil \cdot ds \right|$$
$$= w \cdot \tilde{G}_n(x, t).$$

That is,

$$|E_n(x, t)| \leqslant w \cdot \tilde{G}_n(x, t)$$

for $x \leqslant t$ and n odd.

The last inequality is also true for $x \geqslant t$, all $n \geqslant 1$, and for $n = 0$. Thus we have established that

$$|E_n(x, t)| \leqslant w \cdot \tilde{G}_n(x, t),$$

all $x \in [a, b]$, $n \geqslant 0$.

Using inequality (19) from Lemma 6 we obtain

$$|E_n(x, t)| \leqslant w \cdot \max\left\{\frac{\tilde{G}_n(b, t)}{b-t}, \frac{\tilde{G}_n(a, t)}{t-a}\right\} \cdot |x-t|, \tag{20}$$

all $x \in [a, b]$, fixed $t \in (a, b)$, $n \geqslant 0$.

An integration of inequality (20) with respect to μ produces inequality (6).

Inequality (7) is established from

$$\left|\int_{[a,b]} f\, d\mu - f(t)\right| \leqslant \left|\int_{[a,b]} (f(x)-f(t)) \cdot \mu(dx)\right| + |m-1|\, |f(t)|$$

and

$$(f(x)-f(t)) = \sum_{i=1}^{n} L_i f(t) \cdot g_i(x, t) + L_{n+1} f(t) \cdot N_n(x, t) + E_n(x, t). \quad \blacksquare$$

Proof of Theorem 1′. Since

$$\lim_{N \to +\infty} (g_n(x, s) f_N(|s-t|)) = g_n(x, s) \left\lceil \frac{|s-t|}{h} \right\rceil w,$$

by the bounded convergence theorem we get

$$\lim_{N \to +\infty} \int_t^x g_n(x, s) f_N(|s-t|)\, ds = \int_t^x g_n(x, s) \left\lceil \frac{|s-t|}{h} \right\rceil w\, ds.$$

Thus

$$\lim_{N \to +\infty} \left|\int_t^x g_n(x, s) f_N(|s-t|)\, ds\right| = w \left|\int_t^x g_n(x, s) \left\lceil \frac{|s-t|}{h} \right\rceil ds\right|,$$

i.e.,

$$\lim_{N \to +\infty} \tilde{G}_{nN}(x, t) = w \tilde{G}_n(x, t).$$

Setting

$$G_{nN}(x, t) = \int_t^x g_n(x, s) f_N(|s-t|)\, ds,$$

we have for n odd that

$$\tilde{G}_{nN}(x, t) = G_{nN}(x, t) > 0, \qquad x \neq t,$$

and for n zero or even that

$$\tilde{G}_{nN}(x, t) = \begin{cases} -G_{nN}(x, t) > 0, & a \leqslant x < t, \\ G_{nN}(x, t) > 0, & t < x \leqslant b. \end{cases}$$

In particular $\tilde{G}_{nN}(t, t) = G_{nN}(t, t) = 0$, all $n \geqslant 0$.

Let $n \geqslant 1$. From [8, p. 132(6)] we have

$$\frac{\partial^i g_n(t, t)}{\partial x^i} = \begin{cases} 0, & i = 0, 1, ..., n-1 \\ 1, & i = n. \end{cases}$$

Applying Leibnitz's formula repeatedly, we find that

$$\frac{\partial^i}{\partial x^i} G_{nN}(x, t) = \int_t^x \frac{\partial^i g_n(x, s)}{\partial x^i} f_N(|s-t|)\, ds, \qquad i = 0, 1, ..., n,$$

and

$$\frac{\partial^{n+1}}{\partial x^{n+1}} G_{nN}(x, t) = \int_t^x \frac{\partial^{n+1} g_n(x, s)}{\partial x^{n+1}} f_N(|s-t|)\, ds + f_N(|x-t|),$$
$$\text{all} \quad x \in [a, b].$$

Hence

$$\frac{\partial^i}{\partial x^i} G_{nN}(t, t) = 0, \qquad i = 0, 1, ..., n+1.$$

And one can easily see that

$$\frac{\partial^i}{\partial x^i} \tilde{G}_{nN}(t, t) = 0, \qquad i = 0, 1, ..., n+1.$$

Since L_i is a linear differential operator of order i, $i = 1, ..., n+1$, $L_0 f(t) = f(t)$ (see (1)), we get $L_i \tilde{G}_{nN}(t, t) = 0$, $i = 0, 1, ..., n+1$, $n \geqslant 0$.

From [8, p. 132] we have

$$L_{n+1} G_{nN}(x, t) = f_N(|x-t|), \qquad \text{all} \quad x \in [a, b], \quad n \geqslant 0.$$

Hence for n odd we find that

$$L_{n+1} \tilde{G}_{nN}(x, t) = f_N(|x-t|), \qquad \text{all} \quad x \in [a, b].$$

And for n zero or even we find that

$$L_{n+1} \tilde{G}_{nN}(x, t) = \begin{cases} -f_N(t-x), & a \leqslant x \leqslant t; \\ f_N(x-t), & t \leqslant x \leqslant b. \end{cases}$$

Now consider case (i) of our theorem with f and μ as described in the statements theoreof. Note that

$$L_{n+1}f(x)=\begin{cases} f_N(x-t), & t\leqslant x\leqslant b; \\ 0, & a\leqslant x\leqslant t. \end{cases}$$

Hence one can easily see that

$$\omega_1(L_{n+1}f, h)\leqslant w$$

and $L_i f(t)=0$, $i=0, 1, ..., n+1$.

Consequently the left-hand sides of inequalities (6) and (7) equal $(\tilde{G}_{nN}(b, t)/b-t)d$ which, as $N\to+\infty$, converges to $w(\tilde{G}_n(b, t)/b-t)d$, i.e., to the right-hand side of these inequalities.

Finally consider case (ii) of our theorem with f and μ as described in the statement thereof. Note that

$$L_{n+1}f(x)=\begin{cases} 0, & t\leqslant x\leqslant b; \\ (-1)^{n+1}f_N(t-x), & a\leqslant x\leqslant t. \end{cases}$$

Again one can easily see that

$$\omega_1(L_{n+1}f, h)\leqslant w$$

and

$$L_i f(t)=0, \qquad i=0, 1, ..., n+1.$$

Consequently the left-hand sides of inequalities (6) and (7) equal $(\tilde{G}_{nN}(a, t)/t-a)d$ which, as $N\to+\infty$, converges to $w(\tilde{G}_n(a, t)/t-a)d$, i.e., to the right-hand side of these inequalities. ∎

Proof of Theorem 2. From [8, p. 138, Theorem II] we have

$$f(x)=\sum_{i=0}^{n} L_i f(t)\, g_i(x, t)+\int_t^x g_n(x, s)\cdot L_{n+1}f(s)\cdot ds,$$

all $x, t\in[a, b]$, $n\geqslant 0$.

And from (3) we see that

$$f(x)=\sum_{i=0}^{n} L_i f(t)\cdot g_i(x, t)+L_{n+1}f(t)\cdot N_n(x, t)$$
$$+\int_t^x g_n(x, s)\cdot(L_{n+1}f(s)-L_{n+1}f(t))\cdot ds.$$

Thus

$$E_n(x, t)=\int_t^x g_n(x, s)\cdot(L_{n+1}f(s)-L_{n+1}f(t))\cdot ds, \qquad n\geqslant 0.$$

Since $\omega_1(L_{n+1}f, h) \leqslant w$ (from [1, p. 251, Corollary 2.2]) we have

$$|L_{n+1}f(s) - L_{n+1}f(t)| \leqslant w \cdot \left\lceil \frac{|s-t|}{h} \right\rceil,$$

where $\lceil \cdot \rceil$ is the ceiling of the number.

Let $x \leqslant t$ and n even. Then

$$\begin{aligned}
|E_n(x, t)| &= \left| \int_x^t g_n(x, s) \cdot (L_{n+1}f(s) - L_{n+1}f(t)) \cdot ds \right| \\
&\leqslant \int_x^t (-g_n(x, s)) \cdot |L_{n+1}f(s) - L_{n+1}f(t)| \cdot ds \\
&\leqslant w \cdot \int_x^t g_n(x, s) \cdot \left\lceil \frac{|s-t|}{h} \right\rceil \cdot ds \\
&\leqslant w \cdot \left\lceil \frac{|x-t|}{h} \right\rceil \cdot \left| \int_t^x g_n(x, s)\, ds \right|.
\end{aligned}$$

That is,

$$|E_n(x, t)| \leqslant w \cdot \left\lceil \frac{|x-t|}{h} \right\rceil \cdot |N_n(x, t)|$$

for $x \leqslant t$ and n even.

Let $x \leqslant t$ and n odd. Then

$$\begin{aligned}
|E_n(x, t)| &= \left| \int_x^t (-g_n(x, s)) \cdot (L_{n+1}f(s) - L_{n+1}f(t)) \cdot ds \right| \\
&\leqslant \int_x^t (-g_n(x, s)) \cdot |L_{n+1}f(s) - L_{n+1}f(t)| \cdot ds \\
&\leqslant w \cdot \int_x^t (-g_n(x, s)) \cdot \left\lceil \frac{|s-t|}{h} \right\rceil \cdot ds \\
&\leqslant w \cdot \left\lceil \frac{|x-t|}{h} \right\rceil \cdot \left| \int_t^x g_n(x, s)\, ds \right|.
\end{aligned}$$

That is,

$$|E_n(x, t)| \leqslant w \cdot \left\lceil \frac{|x-t|}{h} \right\rceil \cdot |N_n(x, t)|$$

for $x \leqslant t$ and n odd.

The last inequality is also true for $x \geqslant t$, all $n \geqslant 1$, and for $n = 0$. We have thus established that

$$|E_n(x, t)| \leqslant w \cdot \left\lceil \frac{|x-t|}{h} \right\rceil \cdot |N_n(x, t)|, \tag{21}$$

all $x, t \in [a, b]$, $n \geqslant 0$.

Integrating inequality (21) with respect to μ ($\mu([a, b]) = m$) we get

$$\begin{aligned}
&\int_{[a,b]} |E_n(x, t)| \, \mu(dx) \\
&\quad \leqslant w \cdot \int_{[a,b]} \left\lceil \frac{|x-t|}{h} \right\rceil \cdot |N_n(x, t)| \cdot \mu(dx) \\
&\quad \leqslant w \cdot \int_{[a,b]} \left(1 + \frac{|x-t|}{h}\right) \cdot |N_n(x, t)| \cdot \mu(dx) \\
&\quad = w \cdot \left[\int_{[a,b]} |N_n(x, t)| \cdot \mu(dx) + \frac{1}{h} \cdot \int_{[a,b]} |x-t| \cdot |N_n(x, t)| \cdot \mu(dx)\right] \\
&\quad \leqslant w \cdot \left[m^{1/(n+2)} + \frac{1}{h} \cdot \left(\int_{[a,b]} |x-t|^{n+2} \mu(dx)\right)^{1/(n+2)}\right] \\
&\qquad \cdot \left(\int_{[a,b]} |N_n(x, t)|^{(n+2/n+1)} \cdot \mu(dx)\right)^{(n+1/n+2)} \\
&\quad = w \cdot (m^{1/(n+2)} + 1) \cdot \left(\int_{[a,b]} |N_n(x, t)|^{(n+2/n+1)} \cdot \mu(dx)\right)^{(n+1/n+2)}.
\end{aligned}$$

The last inequality and equality are obtained by applying Hölder's inequality twice and by the choice of h (see (11)), respectively. Therefore we have proved inequality (12).

Inequality (13) is established as follows:

$$\begin{aligned}
\left|\int_{[a,b]} f(x) \, \mu(dx) - f(t)\right| \leqslant |f(t)| \cdot &\left|\int_{[a,b]} g_0(x, t) \, \mu(dx) - 1\right| \\
&+ \sum_{i=1}^{n} |L_i f(t)| \cdot \left|\int_{[a,b]} g_i(x, t) \, \mu(dx)\right| \\
&+ |L_{n+1} f(t)| \cdot \left|\int_{[a,b]} N_n(x, t) \, \mu(dx)\right| \\
&+ \int_{[a,b]} |E_n(x, t)| \, \mu(dx); \qquad L_0 f(t) = f(t). \quad \blacksquare
\end{aligned}$$

Proof of Theorem 3. As in the proof of Theorem 2 we have

$$E_n(x, t) = \int_t^x g_n(x, s) \cdot (L_{n+1} f(s) - L_{n+1} f(t)) \cdot ds,$$

all $x, t \in [a, b]$, $n \geqslant 0$.

The Lipschitz condition $\omega_1(L_{n+1} f, \delta) \leqslant A\delta^\alpha$ implies that

$$|L_{n+1} f(s) - L_{n+1} f(t)| \leqslant A\,|s-t|^\alpha, \qquad \text{all} \quad s, t \in [a, b].$$

Let $x \leqslant t$ and n zero or even. Then

$$\begin{aligned} |E_n(x, t)| &= \left| \int_x^t g_n(x, s) \cdot (L_{n+1} f(s) - L_{n+1} f(t)) \cdot ds \right| \\ &\leqslant \int_x^t g_n(x, s) \cdot |L_{n+1} f(s) - L_{n+1} f(t)| \cdot ds \\ &\leqslant A \int_x^t g_n(x, s)\,|s-t|^\alpha\, ds \leqslant A\,|x-t|^\alpha \left| \int_t^x g_n(x, s)\, ds \right|. \end{aligned}$$

That is,

$$|E_n(x, t)| \leqslant A\,|x-t|^\alpha\,|N_n(x, t)|$$

for $x \leqslant t$ and n zero or even.

Let $x \leqslant t$ and n odd. Then

$$\begin{aligned} |E_n(x, t)| &= \left| \int_x^t (-g_n(x, s)) \cdot (L_{n+1} f(s) - L_{n+1} f(t)) \cdot ds \right| \\ &\leqslant \int_x^t (-g_n(x, s)) \cdot |L_{n+1} f(s) - L_{n+1} f(t)| \cdot ds \\ &\leqslant A \int_x^t (-g_n(x, s))\,|s-t|^\alpha\, ds \leqslant A\,|x-t|^\alpha \left| \int_t^x g_n(x, s)\, ds \right|. \end{aligned}$$

That is,

$$|E_n(x, t)| \leqslant A\,|x-t|^\alpha\,|N_n(x, t)|$$

for $x \leqslant t$ and n odd.

The last inequality is also true for $x \geqslant t$, all $n \geqslant 0$. Thus we have established that

$$|E_n(x, t)| \leqslant A\,|x-t|^\alpha\,|N_n(x, t)|, \tag{22}$$

all $x, t \in [a, b]$, $n \geqslant 0$.

Integrating inequality (22) with respect to μ ($\mu([a, b]) = m$) we get

$$\int_{[a,b]} |E_n(x, t)| \, \mu(dx)$$

$$\leqslant A \int_{[a,b]} |x - t|^{\alpha} \cdot |N_n(x, t)| \cdot \mu(dx)$$

$$\leqslant A \cdot D_\alpha(t) \cdot \left(\int_{[a,b]} |N_n(x, t)|^{(n+2/n+1)} \cdot \mu(dx) \right)^{(n+1/n+2)},$$

where

$$D_\alpha(t) = \left(\int_{[a,b]} |x - t|^{\alpha(n+2)} \cdot \mu(dx) \right)^{1/(n+2)}.$$

The last inequality is a consequence of Hölder's inequality. That is, we have obtained that

$$\int_{[a,b]} |E_n(x, t)| \, \mu(dx)$$

$$\leqslant A \cdot D_\alpha(t) \cdot \left(\int_{[a,b]} |N_n(x, t)|^{(n+2/n+1)} \cdot \mu(dx) \right)^{(n+1/n+2)}. \tag{23}$$

Case of $m \leqslant 1$. By Lemma 7, since $\alpha(n+2) \leqslant n+2$, we have

$$\left(\int_{[a,b]} |x - t|^{\alpha(n+2)} \cdot \mu(dx) \right)^{1/\alpha(n+2)} \leqslant \left(\int_{[a,b]} |x - t|^{n+2} \cdot \mu(dx) \right)^{1/(n+2)};$$

that is,

$$D_\alpha(t) \leqslant h^\alpha. \tag{24}$$

Now the first part of inequality (16) is established by (23) and (24).

Case of $m \geqslant 1$. We observe that

$$\left(\int_{[a,b]} |x - t|^{\alpha(n+2)} \cdot \mu(dx) \right)^{1/\alpha(n+2)}$$

$$= m^{1/\alpha(n+2)} \cdot \left(\int_{[a,b]} |x - t|^{\alpha(n+2)} \cdot \frac{\mu}{m}(dx) \right)^{1/\alpha(n+2)}$$

$$\leqslant m^{1/\alpha(n+2)} \cdot \left(\int_{[a,b]} |x - t|^{n+2} \cdot \frac{\mu}{m}(dx) \right)^{1/(n+2)}$$

$$= m^{((1-\alpha)/\alpha(n+2))} \cdot h.$$

Here we used again Lemma 7. That is, we get

$$D_\alpha(t) \leqslant m^{(1-\alpha/n+2)} \cdot h^\alpha. \tag{25}$$

Finally, inequalities (23) and (25) imply the second part of inequality (16). ∎

References

1. G. A. Anastassiou, A study of positive linear operators by the method of moments, one-dimensional case, *J. Approx. Theory* **45** (1985), 247–270.
2. W. Feller, "An Introduction to Probability Theory and Its Applications," Vol. II, Wiley, New York, 1971.
3. H. H. Gonska, On approximation of continuously differentiable functions by positive linear operators, *Bull. Austral. Math. Soc.* **27** (1983), 73–81.
4. S. Karlin and W. J. Studden, "Tchebycheff Systems: With Applications in Analysis and Statistics," Interscience, New York, 1966.
5. P. P. Korovkin, "Linear Operators and Approximation Theory," Hindustan Publ. Corp., Delhi, India, 1960.
6. B. Mond and R. Vasudevan, "On approximation by linear positive operators," *J. Approx. Theory* **30** (1980), 334–336.
7. O. Shisha and B. Mond, The degree of convergence of sequences of linear positive operators, *Proc. Nat. Sci. U.S.A.* **60** (1968), 1196–1200.
8. D. V. Widder, "A generalization of Taylor's series," *Trans. Amer. Math. Soc.* **30** (1928), 126–154.